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Effective Lagrangian and pair production in cosmology

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Abstract. An effective Lagrangian for the conformally coupled Klein–Gordon field in an expanding 3-flat Robertson–Walker universe is deduced, and from this the number density of the particle pairs created is calculated exactly. An approximation formula for the imaginary part of the effective action density is given. It is applied to the already solved case and to an as yet not exactly solved one.

1. Introduction

The papers of Candelas and Raine (1975) and Dowker and Critchley (1976a, b) are the only ones which are known to the author where for a cosmological gravitational field—the de Sitter universe—the effective Lagrange formalism is applied to the pair creation phenomenon with rigorous results although they get no particle creation out of the vacuum. The reason is, that for the most physical cosmologies, the 'big bang' cosmologies, it is not possible to construct effective Lagrangians and Green's functions which are unique and, in connection with this, enough modes which can be identified with 'particles'. In stressing the modal character, we restrict ourselves in this paper to expansion laws in which these problems do not occur. Because we are interested only in the number of created particles and not in the effective energy–momentum tensor for the vacuum fluctuations, only the imaginary part of the effective action is important to us. The calculation of this part does not involve regularisation operations so we are not forced to insist on manifest covariance in the course of our calculations.

The cosmological gravitational field we have chosen for our exact calculations is the one introduced by Audretsch and Schäfer (1978).

With an approximation formula for the imaginary part of the effective action density we will approximately solve the already solved problem and then an as yet not exactly solved one.

The termini particles are used in the same sense as in Schäfer (1978), i.e. as modes with the 'strongest' WKB-character appearing asymptotically. The notation used is as follows: metric signature is denoted by $(+---)$; $\eta_{\mu\nu}$ denotes the Minkowski metric; ∂_μ equals $\partial/\partial x^\mu$; ∇_μ is the covariant derivative; $g=\det(g_{\mu\nu})$; R is the curvature scalar, Greek indices run from 0 to 3; Latin indices from 1 to 3 and $\hbar=c=1$.

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2. Cosmological fields

The cosmological fields we have chosen for our calculations are as follows ($-\infty < x^\mu < +\infty$):

$$ds^2 = \Omega^2 \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1}$$

$$\Omega^2 = a^2 + b^2\eta^2, \tag{1a}$$

$$\Omega^2 = a^2 + (b^2\eta^2)^2, \tag{1b}$$

in which a^2 and b^2 are constants and $\eta = x^0$. In the case when a^2 equals 0 or is negligible compared to $b^2\eta^2$ or $(b^2\eta^2)^2$ equations (1a) and (1b) describe contracting and expanding radiation and incoherent matter dominated 3-flat Robertson–Walker universes respectively.

3. Quantum field

The quantum field we discuss is the conformally coupled complex Klein–Gordon field Φ with mass m and action S:

$$S = \int \Phi^* F \Phi\, d^4x; \qquad F = -\sqrt{-g}(g^{\mu\nu}\nabla_\mu\nabla_\nu + R/6 + m^2). \tag{2}$$

The equation of motion reads

$$F\Phi = 0. \tag{3}$$

To be able to perform exact calculations we make the same substitution as Raine and Winlove (1975): $\Phi = (-g)^{-1/8}\phi$ and $F = (-g)^{1/8} f (-g)^{1/8}$. We can then write for equations (2) and (3), using expression (1):

$$S = \int \phi^* f \phi\, d^4x, \qquad f = -\eta^{\mu\nu}\,\partial_\mu\,\partial_\nu - m^2\Omega^2 \tag{2a}$$

$$f\phi = 0. \tag{3a}$$

Green's function G for the equation (3a) satisfies the relation

$$f(x)G(x, x') = \delta^4(x - x'). \tag{4}$$

After introducing Schwinger's fictitious Hilbert space it is possible to rewrite equation (4) as an operator equation:

$$\boldsymbol{f}\boldsymbol{G} = \mathbf{1}; \qquad \boldsymbol{f} = \eta^{\mu\nu}\boldsymbol{p}_\mu\boldsymbol{p}_\nu - m^2\boldsymbol{\Omega}^2. \tag{4a}$$

Hence the following relations hold:

$$G(x, x') = \langle x|\boldsymbol{G}|x'\rangle, \qquad f(x)\delta^4(x - x') = \langle x|\boldsymbol{f}|x'\rangle,$$

$$\langle x|\mathbf{1}|x'\rangle = \langle x|x'\rangle = \delta^4(x - x'), \qquad \boldsymbol{x}^\mu|x\rangle = x^\mu|x\rangle,$$

$$[\boldsymbol{x}^\mu, \boldsymbol{x}^\nu] = [\boldsymbol{p}_\mu, \boldsymbol{p}_\nu] = 0, \qquad [\boldsymbol{p}_\nu, \boldsymbol{x}^\mu] = \mathrm{i}\delta^\mu_\nu\mathbf{1}.$$

The formal solution of equation (4a) with Feynman's boundary condition reads

$$\boldsymbol{G}_\infty = \frac{\mathbf{1}}{\boldsymbol{f} + \mathrm{i}0} = -\mathrm{i}\int_0^\infty \exp(\mathrm{i}\boldsymbol{f}s)\, ds. \tag{5}$$

If we define $\langle x, s|x', 0\rangle = \langle x|\exp(-\mathrm{i}\boldsymbol{H}s)|x'\rangle$ with $\boldsymbol{H} = -f$, then the transition amplitude $\langle x, s|x', 0\rangle$ satisfies the Schrödinger equation:

$$\mathrm{i}(\partial/\partial s)\langle x, s|x', 0\rangle = H(x)\langle x, s|x', 0\rangle \tag{6}$$

and the boundary condition $\langle x, 0|x', 0\rangle = \delta^4(x-x')$. The Heisenberg equations of motion corresponding to equation (6) are

$$\mathrm{d}\boldsymbol{x}^\mu/\mathrm{d}s = \mathrm{i}[\boldsymbol{H}, \boldsymbol{x}^\mu], \qquad \mathrm{d}\boldsymbol{p}_\mu/\mathrm{d}s = \mathrm{i}[\boldsymbol{H}, \boldsymbol{p}_\mu]. \tag{7}$$

With the help of the amplitude $\langle x', s|x, 0\rangle$, Green's function $G_\infty(x', x)$ according to equation (5) can be written as

$$G_\infty(x, x') = -\mathrm{i}\int_0^\infty \langle x, s|x', 0\rangle\, \mathrm{d}s. \tag{5a}$$

If we follow Schwinger (1951) and DeWitt (1975) the following representation for the one-loop action functional or the effective action $W^{(1)}$ (which is defined by $\langle\text{out vac}|\text{in vac}\rangle = \mathrm{e}^{\mathrm{i}W^{(1)}}$) is possible:

$$W^{(1)} = \int L(x)\, \mathrm{d}^4x \tag{8}$$

with

$$L(x) = -\mathrm{i}\int_0^\infty s^{-1}\langle x, s|x, 0\rangle\, \mathrm{d}s \tag{9}$$

whereby $L(x)$ is known as the effective Lagrangian. For the cosmological situations (1a) and (1b) $|\text{in vac}\rangle$ and $|\text{out vac}\rangle$ are the vacuum states in both time-asymptotic regions. The probability, that the vacuum has remained the vacuum, is simply given by $\mathrm{e}^{-2\,\mathrm{Im}\,W^{(1)}}$.

4. Pair creation probability

Before we are able to calculate $2\,\mathrm{Im}\,W^{(1)}$ according to equations (8) and (9) we must solve equation (6). This we will do at present for the expansion law (1a). Because in this case the Hamiltonian operator $\boldsymbol{H}$ has the same form as an operator for the usual quantum mechanical free particle and the harmonic oscillator with imaginary frequency, the solution of equation (6) is easily obtained. We have (cf., for example, Feynman and Hibbs 1965)

$$\langle x, s|x', 0\rangle = \frac{-\mathrm{i}}{16\pi^2}\left(\frac{2bms}{\sinh(2bms)}\right)^{1/2} s^{-2}\exp(-\mathrm{i}m^2a^2s)\exp[(-\mathrm{i}/4s)(x^j - x'^j)(x^l - x'^l)\eta_{jl}]$$
$$\times\exp\{(-\mathrm{i}bm/4)[\coth(bms)(x^0 - x'^0)^2 + \tanh(bms)(x^0 + x'^0)^2]\} \tag{10}$$

and

$$\langle x, s|x, 0\rangle = \frac{-\mathrm{i}}{16\pi^2}\left(\frac{2bms}{\sinh(2bms)}\right)^{1/2} s^{-2}\exp[-\mathrm{i}m^2a^2s - \mathrm{i}bm\tanh(bms)(x^0)^2]. \tag{10a}$$

The expression (10a) is also obtainable with the help of the paper of Brown and Duff (1975). If we perform the limit of vanishing cosmic expansion ($b \to 0$) we arrive at the flat space–time expression for the amplitude $\langle x, s|x', 0\rangle$ (Schwinger 1951). This is not the case for the situation which is discussed by Chitre and Hartle (1977) as Nariai and Azuma (1978) have shown. Furthermore, it can be shown that with amplitude (10) and referring to the expression (5a) we obtain a Feynman–Green function which is identical with the one which can be built up with the particle wavefunctions obtained by Audretsch and Schäfer (1978) so that our particles are the same as theirs.

Inserting the expression (10a) into equation (9), we find for the effective Lagrangian

$$L(x)=\frac{1}{16\pi^2}\int_0^\infty \mathrm{d}s\, s^{-3}\left(\frac{2bms}{\sin(2bms)}\right)^{1/2}\exp[-m^2a^2s-bm\tan(bms)(x^0)^2], \tag{11}$$

wherein Schwinger's rotation $s\to -\mathrm{i}s$ (Schwinger 1951) has already been performed. For the effective action density $w^{(1)}=W^{(1)}/\Sigma$ ($\Sigma=\int \mathrm{d}^3x$) we find, using equations (11) and (8), the result

$$w^{(1)}=\int_{-\infty}^{+\infty} L(x)\,\mathrm{d}x^0=\frac{1}{16\pi^{3/2}}\int_0^\infty \mathrm{d}s\, s^{-3}\frac{s^{1/2}}{\sin(bms)}\exp(-m^2a^2s). \tag{12}$$

With the help of the relation $1/(bms-\mathrm{i}0)=\mathrm{P}(1/bms)+\pi\mathrm{i}\delta(bms)$ we obtain the imaginary part of expression (12). Our final result, the pair creation probability per unit coordinate 3-space volume, then reads

$$2\,\mathrm{Im}\, w^{(1)}=\frac{(mb)^{3/2}}{8\pi^3}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-5/2}\exp[-n\pi(ma^2/b)]. \tag{13}$$

5. Approximation formula

Starting from the well-known relation

$$W^{(1)}=\mathrm{i}\ln\det(\boldsymbol{G}_\infty^{-1}\boldsymbol{G}_\infty^{(0)})=\mathrm{i}\,\mathrm{Tr}\ln(\boldsymbol{G}_\infty^{-1}\boldsymbol{G}_\infty^{(0)})$$

with $\boldsymbol{G}_\infty^{(0)}$ as Feynman–Green operator for flat space–times ($b\to 0$ in the expansion laws (1a) and (1b)) it can be shown (cf. Brezin and Itzykson 1970) that $2\,\mathrm{Im}\,w^{(1)}$ is approximately (WKB-approximation) given by

$$2\,\mathrm{Im}\, w^{(1)}\simeq\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|\int_{-\infty}^{+\infty}\mathrm{d}x^0\frac{\dot\omega}{2\omega}\exp\left(-2\mathrm{i}\int_0^{x^0}\omega(\eta)\,\mathrm{d}\eta\right)\right|^2 \tag{14}$$

with

$$\omega=(k^2+m^2\Omega^2)^{1/2}\qquad\text{and}\qquad\dot\omega=\mathrm{d}\omega/\mathrm{d}x^0=\mathrm{d}\omega/\mathrm{d}\eta \tag{14a}$$

if the following conditions are satisfied:

$$|\dot\Omega|\ll m\Omega^2 \tag{14b}$$

and, instead of adiabatic switching, the expansion (cf. Schäfer 1978)

$$\tfrac{3}{4}(\dot\omega/\omega^2)^2-\tfrac{1}{2}\ddot\omega/\omega^3\to 0\qquad\text{for }\eta\to\pm\infty. \tag{14c}$$

In equation (14) the expression $-\int_0^{x^0}\omega(\eta)\,\mathrm{d}\eta$ is the time-dependent part of the classical action for a relativistic point particle with mass m. Because of condition (14c) our quasi-classical particle modes in the time-asymptotic regions ($\eta\to\pm\infty$) are the same as the ones used in the paper by Schäfer (1978). They also coincide with the particle modes above. For both expansion laws (1a) and (1b) the condition (14c) is fulfilled.

From the relation (14*b*) it is easily deduced that for our expansion laws (1*a*) and (1*b*) $a^2=0$ is not allowed now. Without restrictions we therefore set $a^2=1$.

At first, we want to apply formula (14) to the expansion law (1*a*). Then we must, according to equation (14), perform the following integration ($z=bm\eta$):

$$\int_{-\infty}^{+\infty} dz \frac{z}{2(k^2+m^2+z^2)} \exp\left[\frac{2\mathrm{i}}{bm}\int_0^z (k^2+m^2+v^2)^{1/2}\,\mathrm{d}v\right]. \tag{15}$$

This we will do in the complex z-plane. An examination of the integrals of expression (15) shows that we have to cut the complex z-plane between the points $z_+=+\mathrm{i}(k^2+m^2)^{1/2}$ and $+\mathrm{i}\infty$ and $z_-=-\mathrm{i}(k^2+m^2)^{1/2}$ and $-\mathrm{i}\infty$, respectively. Furthermore the two branch points $z_\pm$ are also poles for the integrand belonging to $\int_{-\infty}^{+\infty}\mathrm{d}z$. Referring to the steepest-descent method (cf. Brézin and Itzykson (1970)) we can write for expression (15) approximately:

$$\int_\Gamma \mathrm{d}z\left[\exp\left(\frac{4}{3}\frac{\mathrm{i}}{bm}(2z_+)^{1/2}(z-z_+)^{3/2}\right)\Big/4(z-z_+)\right]\exp\left(\frac{2\mathrm{i}}{bm}\int_0^{z_+}(k^2+m^2+v^2)^{1/2}\,\mathrm{d}v\right) \tag{15a}$$

where Γ is the contour on the upper sheet with $\arg(z-z_+)=5\pi/6$ and $\arg(z-z_+)=\pi/6$ and avoiding the point $z=z_+$ as $z=z_+-\mathrm{i}0$. With the substitution $u=\frac{4}{3}\mathrm{i}(2z_+)^{1/2}(z-z_+)^{3/2}/bm$ we find for the integral (15*a*)

$$\begin{aligned}\frac{2}{3}\oint\frac{\mathrm{d}u}{4u}\mathrm{e}^u \exp\left[\frac{2\mathrm{i}}{bm}\int_0^{z_+}(k^2+m^2+v^2)^{1/2}\,\mathrm{d}v\right]\\ =\frac{\pi}{3}\mathrm{i}\exp\left[\frac{2\mathrm{i}}{bm}\int_0^{z_+}(k^2+m^2+v^2)^{1/2}\,\mathrm{d}v\right].\end{aligned} \tag{15b}$$

The u-integration here is taken along a positively oriented path which follows the real axis from $-\infty$ and returns to $-\infty$, enclosing the origin $u=0$. If we substitute expression (15*b*) into equation (14), we get

$$2\,\mathrm{Im}\,w^{(1)}\simeq\frac{\pi^2}{9}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\exp\left[-\frac{4}{bm}\mathrm{Im}\int_0^{z_+}(k^2+m^2+z^2)^{1/2}\,\mathrm{d}z\right] \tag{16}$$

or, with $\int_0^{z_+}(k^2+m^2+z^2)^{1/2}\,\mathrm{d}z=\mathrm{i}\pi(k^2+m^2)/4$ and performing the k-integration,

$$2\,\mathrm{Im}\,w^{(1)}\simeq[(\pi^2/9(mb)^{3/2}/8\pi^3]\exp(-\pi m/b). \tag{17}$$

This result is in nice agreement with the exact expression (13). We do not obtain result (13) exactly in the limit $b\to 0$ as our result is based on our approximation formula (15*a*). For the expansion law (1*a*) with $a^2=1$ our condition (14*b*) implies $2m\gg b$.

Now we apply formula (14) to the expansion law (1*b*). Using the same method as above, instead of formula (16) we obtain the following expression ($z=bm^{1/2}\eta$):

$$2\,\mathrm{Im}\,w^{(1)}\simeq\frac{\pi^2}{9}\cdot 4\int\frac{\mathrm{d}^3k}{(2\pi)^3}\exp\left(-2\,\mathrm{Im}\,A\right)\cos^2(\mathrm{Re}\,A) \tag{18}$$

with

$$A=\frac{2}{bm^{1/2}}\int_0^{z_0}(k^2+m^2+z^4)^{1/2}\,\mathrm{d}z$$

and $z_0 = [(1+\mathrm{i})/\sqrt{2}](k^2+m^2)^{1/4}$ or $\mathrm{Re}\, A = \mathrm{Im}\, A$ and

$$\mathrm{Im}\, A = \sqrt{2}\, \gamma \frac{(k^2+m^2)^{3/4}}{bm^{1/2}} \qquad \text{with} \qquad \gamma = \int_0^1 (1-u^4)^{1/2}\, \mathrm{d}u \leqslant 0{\cdot}9.$$

Taking the approximation $\cos^2(\mathrm{Re}\, A) \to \frac{1}{2}$ and the abbreviations $y = (x^2+\alpha^2)^{3/4} - \alpha^{3/2}$, $x = 2\gamma^{2/3} k/(b^2 m)^{1/3}$ and $\alpha = 2(\gamma m/b)^{2/3}$, it follows from equation (18) that

$$2\, \mathrm{Im}\, w^{(1)} \simeq \frac{\pi^2}{9} \frac{b^2 m}{12(\pi\gamma)^2} \alpha^{3/2}\, \mathrm{e}^{-\alpha^{3/2}} \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-y} [(1+y\alpha^{-3/2})^{4/3} - 1]^{1/2} (1+y\alpha^{-3/2})^{1/3}. \tag{18a}$$

Because our approximation formula (14) is, according to (14b), valid only if $2m^2 \gg b^2$, i.e. $\alpha^2 \gg 1$, the expression (18a), with sufficient precision, becomes

$$2\, \mathrm{Im}\, w^{(1)} \simeq [(\pi^2/9(mb)^{3/2}/2^{5/4}(3\pi\gamma)^{3/2}]\exp(-2^{3/2}\gamma(m/b)) \tag{18b}$$

in which the definition for α has been used.

6. Discussion

Our exact result (13) has been obtained without manifestly covariant working and without applying any regularising procedure. It is completely in agreement with the result obtained by Audretsch and Schäfer (1978): their expression $n_k = \exp[-\pi\, .\, (k^2+m^2a^2)/mb]$ for the mean number of pairs created per unit coordinate 3-space and 3-momentum volume is related to $2\, \mathrm{Im}\, w^{(1)}$ of equation (13) as it must be on general grounds (Damour 1977) as follows:

$$2\, \mathrm{Im}\, w^{(1)} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \nu^{-1} (n_k)^\nu. \tag{19}$$

With respect to our approximate results (17) and (18b) it is interesting to note that both have the same mass-dependence. Heuristically the exponential factor can be made plausible: the probability of finding a virtual particle pair in which both particles are separated by a distance δx is proportional to $\exp(-m\delta x)$; if, however, δx approximately equals r, where r^{-2} means a typical curvature value (in our case b^2), the particles can become real so that a factor $\exp(-\epsilon m r)$, with ϵ approximately equal to 1, has to be expected (cf. for example Woodhouse 1977). The pre-exponential factor $(mb)^{3/2}$ in the expressions (17) and (18b) means that the virtual particle pairs, which are prepared by the gravitational field for pair creation per space-dimension, are proportional to $(mr^{-1})^{1/2}$. All these things are similar to the constant electric field case in which we have instead of r the term m/eE (Brézin and Itzykson 1970). The relevant difference lies in the time-dimension, which in the case of the electric field produces a further factor $(m(eE/m))^{1/2}$. This however leads directly to a particle pair creation rate.

If we take the same value for b in the equations (17) and (18b) then we find that expansion law (1b) is more effective in producing particle pairs than expansion law (1a).

As can be seen from equation (19) in noting $n_k \ll 1$, which is related to equation (14b), in the WKB regime for the total number density of the created particle pairs out of the cosmological vacuum it follows that (compare equation (14))

$$\int [\mathrm{d}^3 k/(2\pi)^3] n_k \simeq 2\, \mathrm{Im}\, w^{(1)}. \tag{19a}$$

In this connection the papers of Audretsch (1979) and Müller *et al* (1978) have to be mentioned as, although different approaches are used, equivalent† WKB approximation formulas for the number density (19*a*) are obtained.

Finally we are not able to make any statements about local vacuum fluctuation effects because our non-manifest covariant calculations do not admit covariant regularisation procedures. They appear to be adapted only to the global particle pair creation phenomenon.

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† If applied to the same expansion laws.